Problem Sheet 1

1.1 Dirac notation

Consider the n-dimensional vector space $\mathcal{H} = \mathbb{C}^n$ where $\mathbb{C}$ is the field of complex numbers. An element $\vec{v} \in \mathcal{H}$ is a column vector with components v_i, $\vec{v} = (v_1, \ldots, v_n)^t$ (t: transpose). The canonical scalar product is defined as

$$(\vec{v}, \vec{w}) = \sum_{i=1}^{n} v_i^* w_i, \tag{1}$$

where $*$ denotes complex conjugation. $\mathcal{H}$ together with this scalar product is an example for a Hilbert space.

The aim of this problem is to learn how some of your linear Algebra knowledge can be written in so-called Dirac notation. You should be familiar with the maths, it's just a new way of writing things!

(a) Show that the scalar product defined in Eq. (1) has the following properties for $\vec{u}, \vec{v}, \vec{w} \in \mathcal{H}$, $a, b \in \mathbb{C}$:

  (i) $(\vec{v}, \vec{w}) = (\vec{w}, \vec{v})^*$,

  (ii) $(\vec{u}, a\vec{v} + b\vec{w}) = a(\vec{u}, \vec{v}) + b(\vec{u}, \vec{w})$,

  (iii) $(a\vec{u} + b\vec{v}, \vec{w}) = a^*(\vec{u}, \vec{w}) + b^*(\vec{v}, \vec{w})$.

(b) Write all equations in (a) in Dirac notation using $\vec{v} \rightsquigarrow |\, v \rangle$ and $(|\, v \rangle, |\, w \rangle) \rightsquigarrow \langle\, v \,|\, w \,\rangle$.

(c) Consider the canonical basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ of $\mathcal{H}$ with

$$\vec{e}_1 = (1, 0, \ldots, 0)^t, \quad \vec{e}_2 = (0, 1, 0, \ldots, 0)^t, \quad \ldots, \quad \vec{e}_n = (0, 0, \ldots, 1)^t.$$

The basis vectors in $\mathcal{B}$ are orthonormal w.r.t. the scalar product in Eq. (1).

  (i) For any $\vec{v} \in \mathcal{H}$, show that the expansion coefficients $c_i \in \mathbb{C}$ in

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{e}_i \quad \text{are given by} \quad c_i = (\vec{e}_i, \vec{v}). \tag{2}$$

  (ii) Write the expansion of $\vec{v}$ and the expression for c_i in Eq. (2) in Dirac notation.

(d) Consider a linear map $A : \mathcal{H} \to \mathcal{H}$ defined by

$$A\vec{e}_j = \sum_{i=1}^{n} a_{ij} \vec{e}_i \quad \text{with} \quad a_{ij} \in \mathbb{C}. \tag{3}$$

  (i) Find the matrix $M_{\mathcal{B}}^{A} \in \mathbb{C}^{n \times n}$ representing A in the basis $\mathcal{B}$. What would be the matrix representation of A if $\mathcal{B}$ were not an orthonormal basis?

  (ii) Show that $a_{ij} = (\vec{e}_i, A\vec{e}_j)$ for the orthonormal basis vectors $\vec{e}_i$. Make use of Dirac notation in the derivation.

  (iii) The adjoint operator $A^\dagger$ is defined via the equation

$$(A\vec{v}, \vec{w}) = (\vec{v}, A^\dagger \vec{w}) \quad \text{for all} \quad \vec{v}, \vec{w} \in \mathcal{H}. \tag{4}$$

  Write Eq. (4) in Dirac notation and find (using Dirac notation) the matrix $\tilde{a} = M_{\mathcal{B}}^{A^\dagger} \in \mathbb{C}^{n \times n}$ representing $A^\dagger$ in the orthonormal basis $\mathcal{B}$.

  (iv) How is $M_{\mathcal{B}}^{A^\dagger}$ related to $M_{\mathcal{B}}^{A}$? Derive a condition on $M_{\mathcal{B}}^{A}$ for A being self-adjoint (Hermitian), i.e., $A = A^\dagger$.

1.2 Observables and Matrix Exponentials

Please use Dirac notation from now on!

(a) Consider the following matrix representation of the operator σ_y in an orthonormal basis $|\,0\rangle$, $|\,1\rangle$,

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find the eigenvalues s_i and eigenvectors $|\,u_i\rangle$ of σ_y ($i \in \{1,2\}$). Show that the eigenkets of σ_y obey the following relations,

$$\sum_{i=1}^{2} |\,u_i\rangle\,\langle u_i\,| = \mathbb{1}_2, \quad \langle\, u_i\,|\,u_j\,\rangle = \delta_{ij}.$$

Is σ_y an Observable?

(b) The exponent of matrix A is defined as

$$e^A = \mathbb{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(A)^n}{n!}. \tag{5}$$

Every analytic function of a diagonal matrix can be computed entrywise. For example, if you want to take the square root of a diagonal matrix you take the square root of each of its diagonal elements.

(i) Show that $U = e^{iA}$ is unitary if A is self-adjoint.

(ii) Any self-adjoint matrix A is unitarily diagonalizable, that is, it can be written as $A = RDR^\dagger$, where D is a diagonal matrix, with the diagonal elements given by the eigenvalues of A, and R is unitary. Show that

$$e^{iAt} = Re^{iDt}R^\dagger.$$

(c) Use the results from (a) and (b) and calculate $e^{i\sigma_y t}$.

(d) Show that $\sigma_y^2 = \mathbb{1}_2$ and combine this result with Eq. (5) in order to calculate $e^{i\sigma_y t}$ again.

1.3 Tensor Products

The state space of a single qubit is spanned by the orthonormal states $|\,0\rangle$, $|\,1\rangle$. Consider the state space of two qubits a and b, $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$.

The expression $|\,x,y\rangle$ describes a ket in the product space $\mathcal{H}$. It is the tensor product of the states $|\,x\rangle$ and $|\,y\rangle$. It is also written as $|\,x\rangle_a\,|\,y\rangle_b$ or $|\,x\rangle_a \otimes |\,y\rangle_b$. Remember that you have to keep the order of the individual state spaces in $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ fixed!

(a) Calculate the state of the total system $|\,\Psi\rangle = |\,\phi,\psi\rangle = |\,\phi\rangle_a \otimes |\,\psi\rangle_b$ in terms of the basis states $|\,x,y\rangle$ with $x,y \in \{0,1\}$ for

$$|\,\phi\rangle = \lambda_0\,|\,0\rangle + \lambda_1\,|\,1\rangle\,, \quad |\,\psi\rangle = \gamma_0\,|\,0\rangle + \gamma_1\,|\,1\rangle\,.$$

(b) Calculate the scalar product $\langle\,\psi,\phi\,|\,\phi,\psi\,\rangle$.