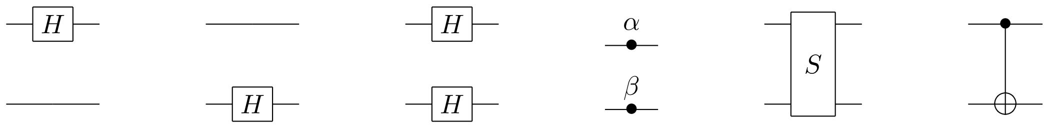
Problem Sheet 2

1.1 Two qubit operations

The circuits below show six unitary operations on the two qubits,

$P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$



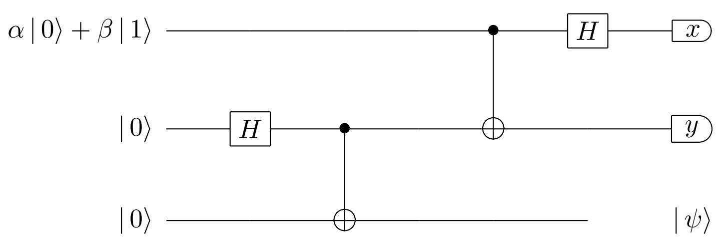
The square root of SWAP matrix has something in common with the square root of NOT.

The first four are described, respectively, by 4×4 unitary matrices which are tensor products $H \otimes \mathbb{1}$, $\mathbb{1} \otimes H$, $H \otimes H$ and $P(\alpha) \otimes P(\beta)$. The matrices of the two remaining gates, known as the square root of SWAP and controlled-NOT, stand out as they do not admit a tensor product decomposition in terms of single-qubit operations. Use the standard tensor product basis, $|\,00\rangle$, $|\,01\rangle$, $|\,10\rangle$, $|\,11\rangle$, and write down unitary matrices for each of the six gates.

1.2 Teleportation

Consider the following quantum network (circuit), containing the Hadamard and the controlled-NOT gates,

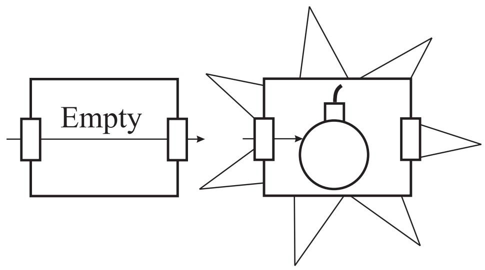
You should remember the action of the Hadamard and the controlled-NOT gates.



The measurement on the first two qubits (counting from the top) gives two binary digits, x and y. The third qubit is not measured. How does the state of the third qubit, $|\,\psi\rangle$, depend on the values x and y? Explain how this can be used to teleport any state of the first qubit to some distant location.

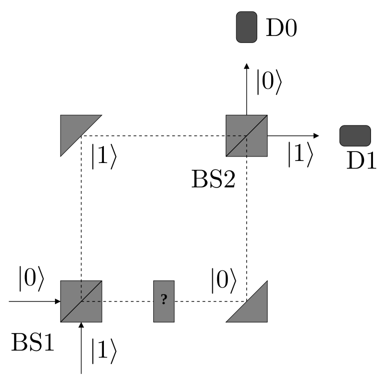
1.3 The Quantum Bomb Tester

You have been drafted by the government to help in the demining effort in a former war-zone. In particular, retreating forces have left very sensitive bombs in some of the sealed rooms. The bombs are configured such that if even one photon of light is absorbed by the fuse (i.e. if someone looks into the room), the bomb will go off. Each room has an input and output port which can be hooked up to external devices. An empty room will let light go from the input to the output ports unaffected, whilst a room with a bomb will explode if light is shone into the input port and the bomb absorbs even just one photon.



Your task is to find a way of determining whether a room has a bomb in it without blowing it up, so that specialised (limited and expensive) equipment can be devoted to defusing that particular room. You would like to know with certainty whether a particular room had a bomb in it.

(a) Consider the setup shown below where the input and output ports are hooked up in the lower arm of a Mach-Zehnder interfereometer.



(i) Consider first an empty room. Write down the transition matrices for one 50 : 50 beam splitter and for two 50 : 50 beam splitters BS1 and BS2. As in the lectures, use the same label for a given input and its transmitted output. What is the output for one photon in the input port $|0\rangle$? Explain the result in physical terms.

(ii) Now assume that the room does contain a bomb. What are the probabilities to detect a photon at one of the detectors D0 or D1 for one photon in the input port $|0\rangle$?

(iii) Design a scheme that allows you - at least part of the time - to decide whether a room has a bomb in it without blowing it up. If you iterate the procedure, what is its overall success rate for the detection of a bomb without blowing it up?

(b) Assume that the beam splitters BS1 and BS2 of the interferometer are different. The probability for transmission of BS1 is $p \ll 1$. BS2 behaves in the opposite way, i.e., its probability for transmission is $1 - p$. Repeat the steps carried out in (a). What is the overall success rate for the detection of a bomb without blowing it up?

(c) Consider a setup of N identical beam splitters (probability of transmission p) where N is an even number. Can you design a scheme (i.e., choose a suitable p) such that the success rate for detecting a bomb without blowing it up approaches 100%?

Write the unitary B for each beam splitter as a rotation matrix $B = e^{i\sigma_x\theta}$, see problem sheet 1.