Problem Sheet 4

1.1 Density matrices, trace and partial trace

(a) A density operator ρ on a finite dimensional Hilbert space is any positive operator $\rho \geq 0$ with trace equal to one. Show that ρ is Hermitian, i.e., $\rho^\dagger = \rho$.

Note that not every Hermitian operator is positive – can you give an example?

(b) Suppose ρ_1 and ρ_2 are density matrices, and let p_1 and p_2 be non-negative numbers such that $p_1 + p_2 = 1$. Show that

$$\rho = p_1\rho_1 + p_2\rho_2$$

is a density matrix. How can you prepare a quantum state described by ρ?

(c) Show that the trace turns an outer product into an inner product, that is, $\mathrm{Tr}(|\, a\rangle \langle b\,|) = \langle\, b\,|\, a\,\rangle$, for any vectors $|\, a\rangle$ and $|\, b\rangle$.

(d) Consider a composite system with state space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Assume that the quantum state of the system is described by the density operator ρ. The matrix elements of the reduced density operator $\rho(1)$ of subsystem $\mathcal{H}_1$ are defined as

$$\langle u_n(1)\,|\,\rho(1)\,|\,u_m(1)\rangle = \sum_p \langle u_n(1), u_p(2)\,|\,\rho\,|\,u_m(1), u_p(2)\rangle\,,$$

where $\{|\,u_n(1)\rangle\}$ ($\{|\,u_n(2)\rangle\}$) is a basis of state space $\mathcal{H}_1$ ($\mathcal{H}_2$). Show that $\rho(1)$ is a density operator.

(e) There are two sources of qubits; one source generates pairs of qubits in state $|\,\psi_1\rangle$ another in state $|\,\psi_2\rangle$, where

$$|\,\psi_1\rangle = \tfrac{1}{2}(|\,00\rangle - |\,01\rangle + |\,10\rangle - |\,11\rangle), \quad |\,\psi_2\rangle = \tfrac{1}{2}(|\,00\rangle + |\,01\rangle + |\,10\rangle - |\,11\rangle).$$

(i) What are the reduced density matrices of individual qubits in each case? Which of the two states is entangled?

(ii) One of the two sources is selected and you are given access to only one member of each pair, say the first qubit, and asked to determine which source the qubits come from. How would you do that?

(iii) Suppose you are still given access to one member of each pair but for each pair it is chosen randomly whether it is the first or the second qubit. Can you still distinguish the two sources?

1.2 Decoherence

A qubit in a pure state $|\,\Psi\rangle = \alpha\,|\,0\rangle + \beta\,|\,1\rangle$ is exposed to decoherence which can be described as

$$|\,0\rangle\,|\,R\rangle \longrightarrow |\,0\rangle\,|\,R_0\rangle\,, \tag{1}$$
$$|\,1\rangle\,|\,R\rangle \longrightarrow |\,1\rangle\,|\,R_1\rangle\,, \tag{2}$$

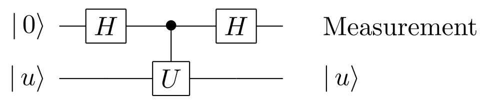
where $|\,R\rangle$, $|\,R_0\rangle$ and $|\,R_1\rangle$ are three normalised states of an environment (not necessarily orthogonal). Show that the density operator of the qubit evolves as,

$$|\,\Psi\rangle\,\langle\Psi\,| \mapsto (1-p)\,|\,\Psi\rangle\,\langle\Psi\,| + p\,\sigma_z\,|\,\Psi\rangle\,\langle\Psi\,|\,\sigma_z \tag{3}$$

and express p in terms of $\langle\, R_0\,|\,R_1\,\rangle$, assuming $\langle\, R_0\,|\,R_1\,\rangle$ is real. This means that with the probability $1-p$ the qubit is not affected by the environment and with the probability p the qubit undergoes the phase-flip error.

1.3 Phase estimation and mixed states

Consider a quantum network represented by the following diagram:



Here the top horizontal line represents a qubit and the bottom one an auxiliary physical system, U is a unitary operation $U \in SU(N)$, and $|u\rangle$ is an eigenvector of U, such that $U|u\rangle = e^{i\phi}|u\rangle$. The measurement on the qubit can be performed either in the $\{|0\rangle, |1\rangle\}$ basis or in the basis $\{|y_+\rangle, |y_-\rangle\}$, where $|y_\pm\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$ are the eigenstates of the Pauli-Y matrix.

(a) What are the probabilities $P_0(\phi)$ and $P_1(\phi)$ that the qubit initially in state $|0\rangle$ will be found respectively in states $|0\rangle$ and $|1\rangle$ at the output if it is measured in the $\{|0\rangle, |1\rangle\}$ basis? What are the corresponding probabilities $P_+(\phi)$ and $P_-(\phi)$ if the qubit is measured in the $\{|y_+\rangle, |y_-\rangle\}$ basis? Suppose you do not know the eigenvalue of $|u\rangle$ but can run the above network many times and perform measurements in the $\{|0\rangle, |1\rangle\}$ and the $\{|y_+\rangle, |y_-\rangle\}$ bases. How would you estimate $\langle u|U|u\rangle$?

(b) Instead of a pure state, $|u\rangle$, the auxiliary system is prepared in a more general state described by the density operator

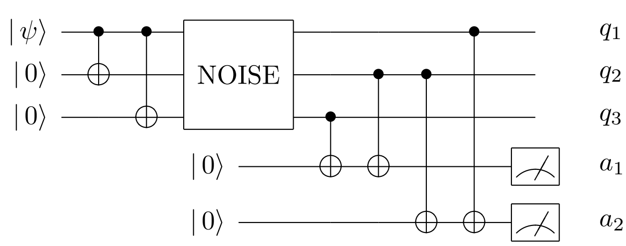
$$\rho = p_1|u_1\rangle\langle u_1| + p_2|u_2\rangle\langle u_2| + \ldots\ldots + p_N|u_N\rangle\langle u_N|, \tag{4}$$

where $\{|u_k\rangle\}$ is an orthonormal set of eigenvectors of U with eigenvalues $e^{i\phi_k}$. How would you estimate $\sum_{k=1}^{N} p_k e^{i\phi_k}$?

(c) Explain how the network above can be used to estimate $\mathrm{Tr}\, U$.

1.4 Error correction

Consider the following circuit. It can be divided into three parts – an encoding step intended to protect the (unknown) quantum state $|\psi\rangle$ against single bit-flip (X) errors, transmission through a noisy channel where a bit-flip occurs independently on each qubit with probability p, and, finally, a circuit that detects any errors and corrects the state, leaving it encoded across the three qubits. Here q_1, q_2 and q_3 label the qubits that form the code and a_1 and a_2 label the auxiliary qubits,



Verify that we can detect a single bit-flip from the noisy channel by measuring the two ancilla qubits (a_1 and a_2) in the computational basis $\{0, 1\}$. Specify what corrections would be required for each possible measurement result.