This Sentence Is False
A Small Collection of Puzzles

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This sentence is false

The probability that this statement is true is at most 50%.
What is the probability that it is true?

It may be customary, in a Festschrift of this kind, to contribute a technical paper — something Kwek and I once laboured over together. And we have certainly given ourselves plenty to choose from, with titles such as *Direct Estimations of Linear and Nonlinear Functionals of a Quantum State*, *Experimental Quantum Multimeter and One-Qubit Fingerprinting in a Single Quantum Device*, and *Robust State Transfer and Rotation Through a Spin Chain via Dark Passage* — titles long enough, one suspects, to count as technical contributions in their own right.

But let us set technicalities aside. There is another, altogether more enjoyable side to our academic life. We do what we do because we like solving problems, and solving problems, happily, does not always require specialised knowledge. The essence of the craft — the hunt for an elegant solution — is often best exercised in what goes by the name of recreational mathematics. All it really demands is a little analytical thinking and a willingness to think sideways.

What follows, then, is a small collection of puzzles and meta-puzzles that Kwek and I have chewed over through the years, mostly over dinner at one hawker stall or another. I would dearly love to credit the original authors, but in most cases I have not the faintest idea who they were. These puzzles have simply drifted into the common folklore — told, retold, embellished, and quietly mutated along the way.

I could not resist illustrating a few of them with AI-generated pictures: an unashamedly modern flourish on some very old favourites. Kwek and I would agree, I think, that as splendid as these new tools are — capable now of not only illustrating the puzzles but cheerfully solving them too — we are rather glad they were not around in our day. We had the pleasure of working out the answers ourselves, and I heartily recommend the same to you. Yes, solutions are provided. No, you should not peek. Sharpen your pencil, pour yourself something agreeable, and enjoy.

The Wizard and the Dwarves

On a remote island, a wizard keeps a group of dwarves as his servants. Each morning at sunrise they assemble in the village square, arranged so that every dwarf can see the face of every other dwarf. Some of them have a large red dot painted on their forehead; the rest do not. Each dwarf can see precisely who among the others is dotted, but has no way of knowing whether he himself is. There are no mirrors, the sea is too rough to throw back a reflection, and the wizard strictly forbids any form of communication — no talking, no gesturing, no signalling of any kind. It is, however, common knowledge among the dwarves that at least one of them bears a dot.

One day the wizard, bored and aware that his dwarves are uncommonly clever, offers them a chance at freedom.

> *"Starting tomorrow morning, and every morning thereafter, I shall ring a bell when you assemble in the square. At the sound of the bell each of you must choose: raise your hand, or keep it down. If no hand goes up, nothing happens and we try again the next day. But the moment at least one hand is raised, the game is over. If every dwarf with a dot — and only those with a dot — has raised his hand, you all go free. A single mistake, and you are mine forever."*

Some days later, every dotted dwarf raised his hand and no other dwarf did. The wizard, true to his word, set them all free. How did the dwarves manage it?

Solution

Although the dwarves never communicate, the passage of silent mornings carries information. Each dwarf reasons about what the others must be thinking, and every quiet day eliminates a possibility.

One dot. Suppose only one dwarf is dotted. He looks around and sees no dots at all. Since it is common knowledge that at least one dot exists, he concludes it must be on his own forehead, and raises his hand on day 1.

Two dots. Suppose two dwarves are dotted. Each sees exactly one dot and thinks: *"If I don't have a dot, then he is the only one — and he will raise his hand on day 1."* When day 1 passes in silence, each realises his hypothesis was wrong: the other dotted dwarf must also have been seeing a dot, which can only be on his own forehead. Both raise their hands on day 2.

Three dots. Each dotted dwarf sees two dots and reasons: *"If I don't have a dot, those two will work it out between them and vote on day 2."* When day 2 passes in silence, each concludes that he too must be dotted. All three raise their hands on day 3.

The general pattern. By induction, if there are n dotted dwarves, each of them sees $n-1$ dots and waits to see whether those $n-1$ dwarves vote on day $n-1$. When they do not, he deduces that they must each be seeing $n-1$ dots themselves — which is only possible if he is dotted too. On day n, all n dotted dwarves reach this conclusion simultaneously and raise their hands.

Why the undotted dwarves stay silent. An undotted dwarf sees n dots and reasons: *"If I also have a dot, then there are $n+1$ dots in all and nothing will happen until day $n+1$. If I don't, there are n and the dotted dwarves will vote on day n."* When day n arrives and hands go up, this is exactly what he expected on the assumption that he is undotted — so he keeps his hand firmly down.

Why common knowledge matters. It is not enough that every dwarf happens to know there is at least one dot. Each dwarf must also know that every other dwarf knows it, and that every dwarf knows that every dwarf knows it, and so on — to arbitrary depth. Without this infinite tower of mutual knowledge, the induction collapses at the very first step. The wizard's public announcement, made before all the dwarves at once, is what establishes this common knowledge and sets the logical clock ticking.

The answer. If there are n dotted dwarves, the vote happens on day n.

A Hundred Ants on a Stick

One hundred ants are placed at random along a thin stick one metre long. At a given instant, each ant chooses a direction and begins to walk at one metre per minute. Whenever two ants collide, both reverse direction and continue at the same speed. An ant that reaches the end of the stick falls off. How long must you wait before you can be certain that every ant has fallen off?

Solution

This little puzzle once helped me crack a rather technical problem involving the quantum transport of identical particles — proof, if any were needed, that recreational mathematics earns its keep. Its beauty lies in becoming trivial the moment you look at it the right way.

Imagine that, instead of bouncing off one another, the ants simply pass through each other when they meet. Since the ants are indistinguishable, the *set* of ant positions on the stick at any moment is the same in both pictures — bouncing and passing — and in particular the moment at which the last ant leaves the stick is the same. But in the passing picture every ant walks in a straight line and reaches the end of the stick in at most one minute. **One minute** is the answer.

Companion Puzzles

If you enjoyed that one, here are two more in the same spirit.

Balls on a line. There are n balls rolling along a line in one direction and k balls rolling along the same line in the opposite direction. The speeds within each group are equal. Initially the two groups are separated, but at some point they begin to collide. The collisions are perfectly elastic. How many collisions occur in total?

Ants on a circle. This one is a little harder. Suppose 24 ants are placed at random on the perimeter of a circle of circumference one metre. Each ant chooses a direction at random and starts walking at one metre per minute, again reversing direction whenever two ants meet. After one minute, what is the probability that Alice, a randomly chosen ant, finds herself back where she started? And would the answer change significantly if there were 25 ants instead of 24?

Hint 1. It helps to imagine that the entire circle is being rotated anticlockwise at one revolution per minute. In this rotating frame, the ants that were moving clockwise stand still, while those moving anticlockwise advance at double speed.

Hint 2. The trick of replacing bouncing with passing tells us only that the *set* of ant locations after one minute is the same as the initial set; a particular ant may well end up at some other ant's starting position. But because ants cannot pass one another, the final configuration must be a rotation of the initial one. The whole collection has been rotated by an integer number of positions, and the question is the probability that this number is a multiple of 24. Convince yourself that if R ants are moving clockwise and L anticlockwise, then after one minute the collection has been rotated clockwise by $R - L$ positions. Rotations by ± 24 correspond to all the ants facing the same way, in which case each walks once around the track without a single collision. This outcome is most unlikely: with each ant choosing a direction independently there are 2^{24} equally likely configurations, of which only two have everyone facing the same way, giving probability $1/2^{23}$. By contrast, $R - L = 0$ — equal numbers going each way — can occur in $\binom{24}{12}$ ways, giving probability $\binom{24}{12}/2^{24} \approx 0.16$.

The Wallet Game

After a few drinks in a bar, a friend of mine suggested a simple game. We each take out our wallets and count the cash. Whoever has more must hand it over to the other, who will then settle the bill.

If I have more money, I ponder, *I shall lose what I have. But if I have less, I shall win more than what I have. So on average, surely, I stand to gain.* Then I notice the smug look on my friend's face and realise that he is reasoning in precisely the same way. Can the game really be favourable to both of us?

Solution

Obviously not. The apparent paradox rests on the implicit assumption that I am equally likely to win or to lose regardless of how much is in my wallet. This is not the case: the more money I have, the more likely I am to lose. The same is true for my friend, and the game turns out to be fair to both.

To make this precise we need a probability distribution over the cash in our wallets. Here is a simple example. I toss a coin and put S$10 in my wallet if it shows heads, S$20 if it shows tails. My friend does the same, and then we play the game. Setting aside the two ties (10–10 and 20–20), I win S$10 in one case and lose S$10 in the other — so the game is fair.

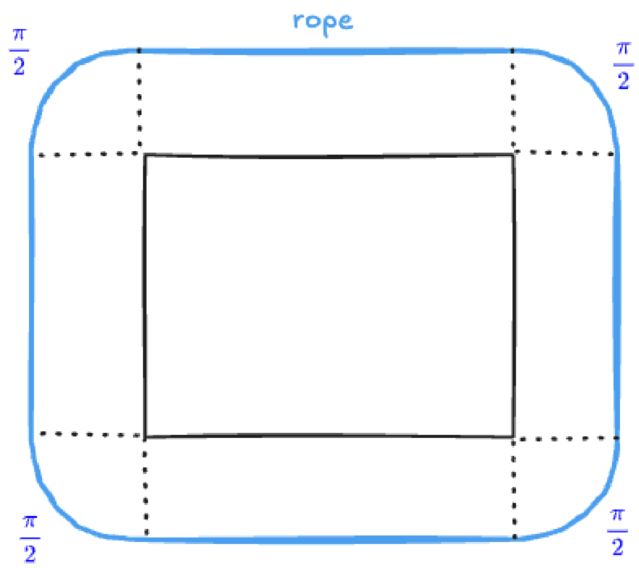
Whiston's Rope

William Whiston (1667–1752) — English theologian, historian, mathematician, and by all accounts a thoroughgoing eccentric — is credited with this charming little puzzle, here in a slightly rephrased form.

A rope some 40,000 kilometres long is wrapped tightly around the equator of the Earth. How much longer would the rope have to be if we wished to lift it one metre off the ground, all the way around the equator?

Solution

The answer is $2\pi(r+1) - 2\pi r = 2\pi$ metres, *regardless of the radius of the Earth.* The same is true for a cubical planet, or a tetrahedral one, or indeed any planet whose cross-section is convex, whatever its shape.



To see why, look at the picture above, which shows a square with a blue rope held one unit away from it at every point. At each corner the rope traces a quarter circle, and the four quarter circles together add exactly 2π units of slack. Now increase the number of corners — try a pentagon, a hexagon, or any convex polygon you please. The exterior angles always sum to 360 degrees, so the fractional arcs traced at the corners, taken together, add up to a full circle. Hence 2π units of slack, every time.

For a smooth convex curve the same reasoning applies in the limit: the corners simply become infinitely many infinitesimal ones, and their contributions still total a single full circle.

Alice, Bob, and the Row of Coins

Fifty coins of various denominations are arranged in a row on a table. Alice picks a coin from one of the ends and slips it into her pocket; Bob then chooses a coin from one of the remaining ends, and the two continue alternating until Bob pockets the last coin. Prove that Alice can play in such a way as to guarantee herself at least as much money as Bob.

Solution

Number the coins from 1 to 50. Observe that no matter how Bob plays, Alice can arrange to capture every even-numbered coin — or, if she prefers, every odd-numbered coin. She simply commits in advance to one of the two parities and takes whichever endpoint coin matches it; the configuration always allows her to do so. Whichever of the two sums (odd-positions or even-positions) is the larger, that is the one she chooses. Bob is left with the smaller.

Comments

Curiously, if there are 51 coins instead of 50, it is usually Bob — the second player — who holds the advantage, despite ending up with fewer coins than Alice. The parity trick suddenly changes hands.

A Pilot's Circuit

A pilot must fly a circuit, visiting a number of airports along the way, each holding a limited supply of aviation gasoline (avgas). The total quantity of avgas across all the airports is exactly enough to complete one full circuit, no more and no less. Is there always an airport at which the pilot, starting with empty tanks, can fuel, take off and make it all the way around?

Solution

Yes, there is — and the argument is rather pretty.

Imagine the pilot sets out with plenty of avgas already in the tanks and flies the entire circuit, topping up at every airport along the way. When she returns to her starting point, the tanks contain exactly the same amount of avgas as when she left, since the total picked up equals the total burned. Somewhere on the route, however, the level in the tanks must reach an absolute minimum — and this minimum must occur at one of the airports, just as the pilot is about to refuel. Starting from *that* airport with empty tanks is then equivalent to flying the same loop with the original level shifted down by the minimum: the tank level never dips below zero, and the pilot completes the circuit.

Alice, Bob, and a Sold-Out Flight

A flight is sold out, and a hundred passengers line up to board. Alice, first in the queue, has mislaid her boarding pass but is waved on board anyway. She picks a seat at random. Each subsequent passenger takes his or her assigned seat if it is free, and otherwise sits in a random unoccupied one. What is the probability that Bob, the very last passenger to board, finds his own seat waiting for him?

Solution

Astonishingly, the answer is exactly 1/2.

When Bob steps on board, only one seat remains free. It must be either his own or the one originally assigned to Alice — for if it had belonged to any of the other 98 passengers, that passenger would have claimed it on entry. So the question reduces to: which of these two seats is the leftover one?

To see that the two are equally likely, consider a slightly modified story. After Alice takes her random seat, every passenger who finds his own seat occupied politely evicts the current occupant — Alice, as we shall see — settles into his rightful place, and sends Alice off to pick another seat at random. In this version, at every stage of boarding, *everyone but Alice* is sitting in their assigned seat, and Alice is wandering between her own seat and the seats of those still to come. When Bob finally boards, only one seat remains: either his own, or Alice's. By symmetry, the two are equally likely, and the probability that Bob finds his seat empty is $1/2$.

It only remains to notice that, since the passengers are indistinguishable from the point of view of the seating arrangement, the set of occupied seats at the end of the modified story is identical to that of the original. The probability is therefore the same: exactly one half.

The Ubiquitous 37%

Forget 42 — it is **37** that quietly explains rather a lot about the universe.

Or, more precisely, *thirty-seven percent.* The formula

$$\left(1 - \tfrac{1}{n}\right)^n \;\longrightarrow\; \frac{1}{e} \;\approx\; 0.37$$

is the patron saint of rare events. Whenever you have a $1/n$ chance of succeeding at something and you try n times — independently, of course — the probability that you *still* haven't succeeded is, for large n, very close to 37%.

The consequences are mildly alarming.

If a large asteroid strikes the Earth roughly once every 50 million years, you have about a 37% chance of dodging the next one over the coming 50 million years. (The other 63% is left as an exercise for the reader.)

If 100 missiles, randomly aimed, fall on a town divided into 100 equal sectors, you should expect about 37 of those sectors to remain pristine. From this you may even hazard an estimate of your adversary's targeting capabilities — though tact may suggest keeping the calculation to yourself.

And if you turn back to *Caesar's Last Breath*, you will find our old friend lurking there too: 37% is precisely the probability that your next inhalation contains *none* of the late dictator's molecules. Which is to say: roughly two breaths in three, you and Caesar are, in a manner of speaking, on speaking terms.

The number 37 is, as it turns out, far more cosmically significant than 42 — and has the additional merit of being true.

Caesar's Last Breath

Can you believe it? With every breath you take, you inhale a molecule or two that Julius Caesar himself exhaled in his dying gasp.

The claim sounds absurd, but it goes back at least to James Jeans, who posed it in his 1940 book on the kinetic theory of gases. Take a single breath. What is the probability that at least one of the molecules you have just drawn in was exhaled by Caesar as he expired on the Ides of March, 44 BC?

Solution

The atmosphere contains roughly 10^{44} air molecules, and a single human breath contains about 10^{22}. Assume — and this is the leap of faith on which the whole argument rests — that after two

millennia of winds, weather and global mixing, Caesar's last 10^{22} molecules are now distributed uniformly throughout the atmosphere.

The probability that any one molecule you inhale is one of Caesar's is therefore

$$\frac{10^{22}}{10^{44}} = 10^{-22},$$

and the probability that it is *not* one of his is $1 - 10^{-22}$. Since you inhale 10^{22} molecules in a breath, and treating them as drawn independently, the probability that *none* of them is from Caesar is

$$\left(1 - 10^{-22}\right)^{10^{22}} \approx \frac{1}{e} \approx 0.37,$$

using the familiar limit $\left(1 - \frac{1}{n}\right)^n \to 1/e$ for large n. The probability of the complementary event — that at least one of Caesar's last molecules is currently making itself at home in your lungs — is therefore about

$$1 - 0.37 \approx 0.63.$$

So roughly two breaths in three connect you, however tenuously, to the dying emperor. The same calculation works equally well for the last breath of Cleopatra, Socrates, or your own great-great-great-grandmother — which is either reassuring or rather alarming, depending on your taste.

A Hundred Lightbulbs

A hundred lightbulbs are arranged in a long row, each with its own switch and all initially off. A hundred children file into the room one after another. The first child flips every switch, turning every bulb on. The second child flips every second switch — bulbs 2, 4, 6, and so on. The third flips every third switch, changing the state of bulbs 3, 6, 9, . . . and so the procession continues until the hundredth child, who flips only the switch of bulb 100.

What is the final state of bulb number 64? And how many of the hundred bulbs are still glowing after the last child has gone?

Solution

Bulb number n is flipped once by every child whose number is a divisor of n. So the bulb ends up *on* if n has an odd number of divisors, and *off* otherwise.

Now, divisors usually come in pairs: for each divisor d of n, the quotient n/d is another. Take bulb 24: its divisors are $\{1, 24\}, \{2, 12\}, \{3, 8\}, \{4, 6\}$ — four neat pairs, eight divisors in all, and so the bulb is flipped an even number of times and ends up off.

The pairing fails only when $d = n/d$, that is, when n is a perfect square. For bulb 64, the divisors $\{1, 64\}, \{2, 32\}, \{4, 16\}$ pair off as usual, but 8 is paired with itself. The eighth child flips the switch only once, leaving the divisor count odd and bulb 64 *on*.

So the bulbs that remain illuminated are precisely those numbered with a perfect square:

$$1, \ 4, \ 9, \ 16, \ 25, \ 36, \ 49, \ 64, \ 81, \ 100.$$

Ten bulbs in all. Bulb 64, in particular, glows triumphantly to the end.

Three Errrors

Consider the following sentence:

> *There are three errrors in this sentence.*

How many errors does it actually contain?

Solution

The first is plain enough: *errrors* is misspelt. So far, one error.

But the sentence claims there are three, and we have found only one — making the claim itself wrong. That is the second error. We are now staring at a sentence that contains exactly two errors and asserts that it contains three, and we are dutifully searching for a third.

The third error is the absence of a third error. The sentence is false in a way that makes its own falsity the very thing it was missing — a small logical knot that tightens the more you pull at it. If we count the missing-third-error as the third, then the sentence has three errors and is true, in which case the count was right after all and there are only two errors, in which case...

It is a close cousin of the liar paradox, *This sentence is false* — the same vertigo, dressed up for an evening out.

Final Word

It is sometimes hard to predict where this kind of recreational puzzling will lead — which is rather the point. Curiosity has a way of taking the long way round.

By way of illustration, Yutaka Nishiyama of Osaka has argued that egg-shaped eggs enjoy a quiet evolutionary advantage over spherical ones. Ovoids do not roll as well as spheres, and a cliffside nest is no place for a sphere. A cube or a tetrahedron would, admittedly, perform even better in this respect — but the difficulties at the laying end of the process are best left to the imagination. Unless, that is, you are a Gillygaloo bird or a fairy shrimp, in which case the matter is apparently in hand.

Nishiyama's analysis can be found in his paper *A Mathematical Approach to the Shape of Eggs*, and a more digestible account in *Easter Eggs for Mathematicians* in *Improbable Research.*

Happy puzzling, Kwek. May the next sixty-five years yield as many hawker dinners, half-solved problems, and unexpected detours as the last.